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The η decay into 3π in asymmetric nuclear medium

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We explore how the η - π^0 mixing angle and the η meson decay into $\pi^+\pi^-\pi^0$ and $3\pi^0$ are modified in the nuclear medium on the basis of the in-medium chiral effective theory with varying isospin asymmetry α , where $\alpha \equiv \delta\rho/\rho$ with $\delta\rho = \rho_n - \rho_p$ and $\rho = \rho_n + \rho_p$. We find that the larger the isospin asymmetry $\delta\rho$ and the smaller the total density ρ , the more enhanced the mixing angle. We show that the decay width in the nuclear medium has an additional density dependence that cannot be renormalized into that of the mixing angle: The additional term originates from the vertex proportional to a low-energy constant c_1 , which only comes into play in the nuclear medium but not in free space. It turns out that the resultant density effect on the decay widths overwhelms that coming from the isospin asymmetry, and the higher the ρ , the more enhanced the decay widths; the width for the $\pi^+\pi^-\pi^0$ decay is enhanced by a factor of two to three at the normal density ρ_0 with a minor increase due to $\delta\rho$, while that for the $3\pi^0$ decay shows only a small increase of around 10% even at ρ_0 . We mention the possible relevance of the partial restoration of chiral symmetry to the unexpected density effect on the decay widths in the nuclear medium.
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Subject Index D33

1. Introduction

Quantitative understanding of η meson decay into three π is a long-standing problem in hadron physics [1–9]. The decay process is prohibited by G parity conservation with isospin symmetry being taken for granted, and the electromagnetic correction is found to vanish in the leading order [1]. The origin of the decay is attributed to the isospin symmetry breaking inherent in quantum chromodynamics (QCD), the small current-quark mass difference between u and d quarks¹. Due to the isospin symmetry breaking, the observed η and π^0 do not correspond to the eigenstate of the flavor SU(3) but to their mixed state,

$$\begin{pmatrix} \eta \\ \pi^0 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \eta_8 \\ \pi_3 \end{pmatrix}, \quad (1)$$

where we denote the mass eigenstate as η and π^0 , and the flavor SU(3) eigenstate as η_8 and π_3 . The angle θ is called the η - π^0 mixing angle. The π_3 component of the η meson enables the meson to decay into 3π , showing that the mixing angle plays an essential role in the decay. However, an analysis based on the current algebra shows a large discrepancy with the experimental result [2].

¹ Regarding this η decay process, S. Weinberg dealt with this process as the $U_A(1)$ problem [3] related to the η mass or π^0 , η , and η' mixing properties, and various attempts were suggested to explain the experimental data; see, e.g., Refs. [10–12].

Recent theoretical development has revealed the significance of the final-state interaction between pions in addition to the u and d quark mass difference for the quantitative account of the experimental data². A phenomenological inclusion of the effect of the final-state interaction [4] shows a good agreement with the observed decay width and it is shown that the higher-order contribution of chiral perturbation theory to the decay process that contains the effect of the final-state interaction is quite large [5,7,8].

The modification of the hadron properties in the environment, which is characterized by temperature, baryonic density, electromagnetic field, and so on, is an interesting topic in hadron physics (see, e.g., Refs. [17,18]). Furthermore, the effects of isospin asymmetry in a nuclear medium on hadron properties are being investigated in various systems, including the pion–nucleus system [19,20], the few-body system of hyper-nuclei [21], and the equation of state of nuclear matter [22,23]. In this paper, we show that the η decay into three pions provides yet another example revealing the interesting effects caused by the isospin asymmetry of the nuclear medium.

As for the in-medium properties of the η meson, some intensive searches for η -bound states in a nucleus has been made (for a recent review, see, e.g., Ref. [24]), where the focus is put on the η -optical potential and/or a possible mass shift in the nuclear medium.

In this paper, we study the effect of the asymmetric nuclear medium, focusing on the three- π decay of η in the isospin-asymmetric nuclear medium, where the isospin-breaking background field is present in addition to the different u and d quark masses³. We shall show that the isospin asymmetry in the nuclear medium increases the mixing of the η and π^0 mesons and thereby leads to an enhancement of the decay rates of the η meson to three pions. We shall also find that the (isosymmetric) total baryon density unexpectedly causes an enhancement of the decay width, which is found to be associated with the phenomenon of the partial restoration of chiral symmetry in the nuclear medium.

This paper is organized as follows. In Sect. 2, we introduce the model Lagrangian and explain the calculation of the η – π^0 mixing angle and the decay amplitude of η into three π in the asymmetric nuclear medium in chiral effective field theory. We evaluate the η – π^0 mixing angle in the asymmetric nuclear medium in Sect. 3. Then we discuss the decay amplitude of the η meson decay into 3π in the isospin-asymmetric nuclear medium with numerical results in Sect. 4. A brief summary and concluding remarks are presented in Sect. 5. In the Appendix, we present the explicit forms of the meson–baryon vertices derived from the chiral Lagrangian and used in the calculation in the text.

2. Preliminaries

To investigate the η meson decay into three π in the nuclear medium, we apply the chiral effective field theory in the nuclear medium. A full account of the in-medium chiral perturbation may be seen in Refs. [26,27]. The basic degrees of freedom are the flavor-octet pseudoscalar mesons and baryons.

² In Refs. [13,14], a fairly good agreement is obtained in the linear sigma model analysis. The linear sigma model contains the explicit iso-singlet sigma meson degree of freedom, and the sigma mesons have some relationship with the s-wave $\pi\pi$ correlation. The recent analysis of the experimental data supports the existence of the σ pole in the $\pi\pi$ channel ($I = 0$) [16]. The importance of the two- π correlation in the η – 3π system is discussed in Ref. [4], as mentioned in the text. Here, we note that it is reported in Ref. [6] that the inclusion of the scalar mesons in a different manner hardly affects the η decay width.

³ An early version of the present work was presented in Ref. [25].

Then the chiral Lagrangian needed for our calculation reads [28,29]

$$\mathcal{L} = \mathcal{L}_{\pi\pi}^{(2)} + \mathcal{L}_{\pi\pi}^{(4)} + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)}, \quad (2)$$

$$\mathcal{L}_{\pi\pi}^{(2)} = \frac{f^2}{4} \left\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U \chi^\dagger \right\rangle, \quad (3)$$

$$\begin{aligned} \mathcal{L}_{\pi\pi}^{(4)} = & L_1 \left\langle D_\mu U (D^\mu U)^\dagger \right\rangle^2 + L_2 \left\langle D_\mu U (D_\nu U)^\dagger \right\rangle \left\langle D^\mu U (D^\nu U)^\dagger \right\rangle \\ & + L_3 \left\langle D_\mu U (D^\mu U)^\dagger D_\nu U (D^\nu U)^\dagger \right\rangle + L_4 \left\langle D_\mu U (D^\mu U)^\dagger \right\rangle \left\langle \chi U^\dagger + U \chi^\dagger \right\rangle \\ & + L_5 \left\langle D_\mu U (D^\mu U)^\dagger (\chi U^\dagger + U \chi^\dagger) \right\rangle + L_6 \left\langle \chi U^\dagger + U \chi^\dagger \right\rangle^2 + L_7 \left\langle \chi U^\dagger - U \chi^\dagger \right\rangle^2 \\ & + L_8 \left\langle U \chi^\dagger U \chi^\dagger + \chi U^\dagger \chi U^\dagger \right\rangle^2 - i L_9 \left\langle f_{\mu\nu}^R D^\mu U (D^\nu U)^\dagger + f_{\mu\nu}^L (D^\mu U)^\dagger D^\nu U \right\rangle \\ & + L_{10} \left\langle U f_{\mu\nu}^L U^\dagger f_R^{\mu\nu} \right\rangle + H_1 \left\langle f_{\mu\nu}^R f_R^{\mu\nu} + f_{\mu\nu}^L f_L^{\mu\nu} \right\rangle + H_2 \left\langle \chi \chi^\dagger \right\rangle, \end{aligned} \quad (4)$$

$$\mathcal{L}_{\pi N}^{(1)} = \left\langle \bar{B} \left(i \not{D} - m_N + \frac{g_A}{2} \gamma^\mu \gamma_5 u_\mu \right) B \right\rangle, \quad (5)$$

$$\begin{aligned} \mathcal{L}_{\pi N}^{(2)} = & c_1 \langle \chi_+ \rangle \langle \bar{B} B \rangle - \frac{c_2}{4m_N^2} \langle u_\mu u_\nu \rangle \langle \bar{B} D^\mu D^\nu N + \text{h.c.} \rangle + \frac{c_3}{2} \langle u_\mu u^\mu \rangle \langle \bar{B} B \rangle \\ & - \frac{c_4}{4} \langle \bar{B} \gamma^\mu \gamma^\nu [u_\mu, u_\nu] B \rangle + c_5 \left\langle \bar{B} \left(\chi_+ - \frac{1}{2} \langle \chi_+ \rangle \right) B \right\rangle + \left\langle \bar{B} \sigma^{\mu\nu} \left(\frac{c_6}{2} f_{\mu\nu}^+ + \frac{c_7}{2} v_{\mu\nu}^{(s)} \right) B \right\rangle. \end{aligned} \quad (6)$$

Here,

$$U = \exp \left(i \frac{\pi^a \lambda^a}{f} \right), \quad u = \sqrt{U} = \exp \left(i \frac{\pi^a \lambda^a}{2f} \right) \quad (7)$$

$$\pi = \pi^a \lambda^a = \begin{pmatrix} \pi_3 + \frac{\eta_8}{\sqrt{3}} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi_3 + \frac{\eta_8}{\sqrt{3}} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta_8 \end{pmatrix} \quad (8)$$

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^- & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}, \quad (9)$$

$$\chi = 2B_0 \mathcal{M}, \quad \mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}, \quad (10)$$

$$f_{\mu\nu}^L = \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu], \quad f_{\mu\nu}^R = \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu], \quad (11)$$

$$D_\mu = \partial_\mu + \Gamma_\mu, \quad \Gamma_\mu = \frac{1}{2} \left(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger \right), \quad (12)$$

$$u_\mu = i \left(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger \right), \quad \chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u, \quad (13)$$

$$f_{\mu\nu}^\pm = u f_{\mu\nu}^L u^\dagger \pm u^\dagger f_{\mu\nu}^R u, \quad v_{\mu\nu}^{(s)} = \partial_\mu v_\nu^{(s)} \pm \partial_\nu v_\mu^{(s)}, \quad (14)$$

and $\langle \cdots \rangle$ means the trace in the flavor space. The Lagrangian that determines the interaction between the hadrons is constructed so as to be invariant under the chiral transformation of the hadron fields as

$$U(x) \mapsto RU(x)L^\dagger, \quad (15)$$

$$u(x) \mapsto \sqrt{RU(x)L^\dagger} \equiv Ru(x)K^{-1}(L, R, U), \quad K(L, R, U) = \sqrt{RUL^\dagger}R\sqrt{U}, \quad (16)$$

$$B(x) \mapsto K(L, R, U)B(x)K^\dagger(L, R, U). \quad (17)$$

The parameters f , B_0 , m_i , L_i , H_i , g_A , and c_i appearing in the Lagrangian are low-energy constants (LECs), the values of which cannot be fixed solely from the symmetry and determined phenomenologically; the values that are used in our calculation are presented in Refs. [28,30].

The relevant degrees of freedom of baryon fields in Eq. (9) are proton and neutron because we are interested in the medium modification by the nucleon background. We denote the nucleons in a doublet form as $N = {}^t(p, n)$. Although it is known [31] (see also, e.g., Ref. [24] for a recent review) that the coupling with $N^*(1535)$ resonance contributes to the η self-energy, the incorporation of $N^*(1535)$ and other excited baryons with strangeness is beyond the scope of the present work. We shall later give a brief comment on possible modification of the results due to the coupling with $N^*(1535)$. The meson–baryon vertices are derived by expanding U with respect to the meson fields π^a . The explicit forms of the vertices to be used in our calculation are presented in Appendix A.

The medium effect is contained in the nucleon propagator $iG(p, k_f)$,

$$iG(p, k_f) = (p + m_N) \left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi\delta(p^2 - m_N^2)\delta(p_0)\theta(k_f - |\vec{p}|) \right\}, \quad (18)$$

where m_N and k_f are the nucleon mass and Fermi momentum, respectively: The first term of Eq. (18) is the contribution of the nucleon propagation in free space and the second term accounts for the Pauli blocking effect of the nuclear medium. The number density and the Fermi momentum of the nucleon are related by $\rho_{p,n} = \frac{k_f^{(p,n)3}}{3\pi^2}$. The total baryon density ρ and asymmetric density $\delta\rho$ are defined by $\rho = \rho_n + \rho_p$ and $\delta\rho = \rho_n - \rho_p$, respectively. Note that $\delta\rho > 0$ means that $\rho_n > \rho_p$ in the present definition. The nuclear asymmetry is also defined by $\alpha = \delta\rho/\rho$.

We note that the value of the Fermi momentum k_f around the normal nuclear density $\rho_0 = 0.17 \text{ fm}^{-3}$ is roughly $2m_\pi$. In our calculation, we regard k_f as being as small as the pseudoscalar meson masses and momenta, which are the expansion parameter in the ordinary chiral perturbation theory. We call k_f and the masses of the pseudoscalar mesons small quantities and denote them generically by q in the following. Hence, our calculation is the expansion with respect to the number of mesons or nucleon loops, because these loops supply additional small quantities compared with the tree level. In addition, we regard the nucleon mass m_N as a large enough quantity and neglect the ratios of the other quantities to m_N . Here, we note that the states of the nuclear medium are treated as a Fermi gas in the leading order in the present formalism, and, accordingly, the nucleon–nucleon interaction is switched off initially.

We calculate the η – 3π decay width up to $O(q^5)$ in the leading order of the asymmetric density $\delta\rho$. The final states of the three π can be two patterns, i.e., $\pi^0\pi^+\pi^-$ and three π^0 . The decay amplitude in free space up to $O(q^4)$ is given in Refs. [5,7].

3. The η - π^0 propagator in the asymmetric nuclear medium

This section is devoted to calculation of the η - π^0 propagator and the η - π^0 mixing angle in the asymmetric nuclear medium. Here, we denote η and π^0 as the mesons of the mass eigenstate and η_8 and π_3 as the SU(3) eigenstate; the mass and SU(3) eigenstates are related by Eq. (1).

The η and π^0 propagator in the asymmetric nuclear medium $D(p; k_f)$ reads

$$D^{-1}(p; k_f^{(p,n)}) = \begin{pmatrix} D_{\eta_8}^{(0)-1}(p) - \Pi_{\eta_8}(k_f^{(p,n)}) & -\Pi_{\eta_8\pi_3}(k_f^{(p,n)}) \\ -\Pi_{\eta_8\pi_3}(k_f^{(p,n)}) & D_{\pi_3}^{(0)-1} - \Pi_{\pi_3}(k_f^{(p,n)}) \end{pmatrix}, \quad (19)$$

where the $D_i^{(0)}(p)$ ($i = \pi_3, \eta_8$) are the propagators of the pseudoscalar mesons in the triplet and octet states in free space, respectively, and the $\Pi_i(k_f)$ is the in-medium self-energy. The $\Pi_{\eta_8\pi_3}(k_f)$ is the transition amplitude of the η_8 and π_3 mesons. The meson masses squared, $m_{\eta_8}^2$ and $m_{\pi_3}^2$, are the poles of D_{η_8} and D_{π_3} , respectively, and the off-diagonal term of the η - π^0 mass matrix $m_{\eta_8\pi_3}^2$ is equal to $\Pi_{\eta_8\pi_3}$. The η - π^0 mixing angle θ is obtained in terms of the masses:

$$\tan 2\theta = -\frac{2m_{\eta_8\pi_3}^2}{m_{\eta_8}^2 - m_{\pi_3}^2}. \quad (20)$$

The diagrams that contribute to the η - π^0 mixing are shown in Fig. 1. We denote the contributions from diagrams (i), (ii), and (iii) as $\Pi_{\eta_8\pi_3}^{(i)}$, $\Pi_{\eta_8\pi_3}^{(ii)}$, and $\Pi_{\eta_8\pi_3}^{(iii)}$, respectively. Actually, $\Pi_{\eta_8\pi_3}^{(i)}$ vanishes because the $\eta\pi^0\pi^+\pi^-\bar{N}N$ vertex from $\mathcal{L}_{\pi N}^{(1)}$ is zero, as is shown in Appendix A.2.1.

The $\eta\bar{N}N$ and $\pi^0\bar{N}N$ vertices relevant to $\Pi_{\eta_8\pi_3}^{(ii)}$ are given in Eqs. (A4) and (A5), and we have

$$\begin{aligned} -i\Pi_{\eta_8\pi_3}^{(ii)} = & -\left(-\frac{g_A}{2\sqrt{3}f}\right)\left(-\frac{g_A}{2f}\right)\int\frac{d^4p}{(2\pi)^4}\text{tr}\{\gamma_5\mathbf{k}(\mathbf{k}+\mathbf{p}+m_N)(-\gamma_5\mathbf{k})(\mathbf{p}+m_N)\} \\ & \times\left\{\frac{i}{p^2-m_N^2+i\epsilon}-2\pi\delta(p^2-m_N^2)\theta(p_0)\theta(k_f-|\vec{p}|)\right\} \\ & \times\left\{\frac{i}{(p+k)^2-m_N^2+i\epsilon}-2\pi\delta((p+k)^2-m_N^2)\theta(p_0+k_0)\theta(k_f-|\vec{p}+\vec{k}|)\right\}. \end{aligned} \quad (21)$$

Here, the minus sign on the right-hand side of the first line comes from the nucleon loop, and the tr in the first line means the trace of the gamma matrix. In our calculation, we take the η -rest frame, so the η momentum k is given by $k = (m_\eta, \mathbf{0})$. Eliminating the contribution from the nucleon propagation in free space, we have

$$\begin{aligned} -i\Pi_{\eta_8\pi_3}^{(ii)} = & -\frac{1}{\sqrt{3}}\left(\frac{g_A}{2f}\right)^24(-2\pi i)\int\frac{d^4p}{(2\pi)^4}\left\{-2(k\cdot p)^2+k^2p^2+k^2m_N^2-k^2(k\cdot p)\right\} \\ & \times\left\{\frac{1}{p^2-m_N^2+i\epsilon}\delta((p+k)^2-m_N^2)\theta(p_0+k_0)\theta(k_f-|\vec{p}+\vec{k}|)\right. \\ & \left.+\frac{1}{(p+k)^2-m_N^2+i\epsilon}\delta(p^2-m_N^2)\theta(p_0)\theta(k_f-|\vec{p}|)\right\}. \end{aligned} \quad (22)$$

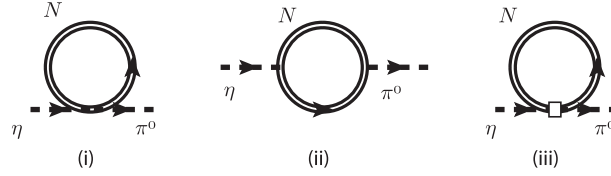


Fig. 1. The diagrams contributing to the η - π^0 mixing angle in the asymmetric nuclear medium. The dashed and double-solid lines represent the meson and nucleon propagations, respectively. The white box at the vertex of diagram (iii) means the $\mathcal{L}_{\pi N}^{(2)}$ -originated vertex.

Changing the integration variable from p to $p' = p + k$ in the first term in the brackets, we obtain

$$\begin{aligned} -i\Pi_{\eta 8\pi 3}^{(ii)} &= -\frac{1}{\sqrt{3}} \left(\frac{g_A}{2f} \right)^2 4(-2\pi i) \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{-2(k \cdot p)^2 + k^2 p^2 + k^2 m_N^2 + (k \cdot p)k^2}{(p-k)^2 - m_N^2 + i\epsilon} \right. \\ &\quad \left. + \frac{-2(k \cdot p)^2 + k^2 p^2 + k^2 m_N^2 - k^2(k \cdot p)}{(p+k)^2 - m_N^2 + i\epsilon} \right\} \delta(p^2 - m_N^2) \theta(p_0) \theta(k_f - |\vec{p}|) \\ &= -\frac{1}{\sqrt{3}} \left(\frac{g_A}{2f} \right)^2 4(-2\pi i) \int \frac{d^3 p}{(2\pi)^4} 2 \times \frac{(k \cdot p)k^2}{-2p \cdot k} \frac{\theta(k_f - |\vec{p}|)}{2E_N(\vec{p})} \\ &= -i \frac{g_A^2 m_\eta^2}{4\sqrt{3} m_N f^2} \rho. \end{aligned} \quad (23)$$

Here, the nucleon mass is treated as a large quantity, and hence the nucleon energy $E_N(\vec{p})$ is approximated by m_N and the initial η meson is at rest. Noting that the π^0 couples to a proton and a neutron with opposite signs, we obtain the final form as

$$-i\Pi_{\eta 8\pi 3}^{(ii)} = i \frac{g_A^2 m_\eta^2}{4\sqrt{3} m_N f^2} \delta \rho. \quad (24)$$

Now, we decompose $\Pi_{\eta 8\pi 3}^{(iii)}$ into $\Pi_{\eta 8\pi 3}^{(iii1)}$ and $\Pi_{\eta 8\pi 3}^{(iii5)}$, which come from the terms proportional to c_1 and c_5 in the chiral Lagrangian in Eq. (6), respectively. The $\eta\pi^0\pi^+\pi^-\bar{N}N$ vertex proportional to c_1 contained in $\mathcal{L}_{\pi N}^{(2)}$ is given in Eq. (A8), and thus we have for $\Pi_{\eta 8\pi 3}^{(iii1)}$

$$\begin{aligned} -i\Pi_{\eta 8\pi 3}^{(iii1)} &= -\left(i \frac{4c_1 m_1^2}{\sqrt{3} f^2} \right) \int \frac{d^4 p}{(2\pi)^4} \text{tr}(\not{p} + m_N) \\ &\quad \times \left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi \delta(p^2 - m_N^2) \theta(p_0) \theta(k_f - |\vec{p}|) \right\}, \end{aligned}$$

which is reduced to

$$-i\Pi_{\eta 8\pi 3}^{(iii1)} = i \frac{4c_1 m_1^2}{\sqrt{3} f^2} \rho. \quad (25)$$

With the use of the $\eta\pi^0\bar{N}N$ vertex coming from the c_5 term in Eq. (A11), $\Pi_{\eta 8\pi 3}^{(iii5)}$ is reduced to

$$\begin{aligned} -i\Pi_{\eta 8\pi 3}^{(iii5)} &= -\left(-i \frac{4c_5 B_0 m_u}{\sqrt{3} f^2} \right) \int \frac{d^4 p}{(2\pi)^4} \text{tr}(\not{p} + m_N) \\ &\quad \times \left[\frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi \delta(p^2 - m_N^2) \theta(p_0) \theta(k_f^{(p)} - |\vec{p}|) \right] \end{aligned}$$

$$\begin{aligned}
 & - \left(i \frac{4c_5 B_0 m_d}{\sqrt{3} f^2} \right) \int \frac{d^4 p}{(2\pi)^4} \text{tr}(\not{p} + m_N) \\
 & \times \left[\frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi \delta(p^2 - m_N^2) \theta(p_0) \theta(k_f^{(n)} - |\vec{p}|) \right] \\
 & - \frac{1}{2} \left(-i \frac{4c_5 m_1^2}{\sqrt{3} f^2} \right) \int \frac{d^4 p}{(2\pi)^4} \text{tr}(\not{p} + m_N) \\
 & \times \left\{ \left[\frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi \delta(p^2 - m_N^2) \theta(p_0) \theta(k_f^{(p)} - |\vec{p}|) \right] \right. \\
 & \left. + \left[\frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi \delta(p^2 - m_N^2) \theta(p_0) \theta(k_f^{(n)} - |\vec{p}|) \right] \right\}, \quad (26)
 \end{aligned}$$

which tells us that the contribution from the nuclear medium is given by

$$\begin{aligned}
 -i \Pi_{\eta 8 \pi^3}^{(\text{iii5})} &= i \frac{4c_5}{2\sqrt{3} f^2} B_0 (m_u + m_d) \delta\rho \\
 &= i \frac{2c_5 m_\pi^2}{\sqrt{3} f^2} \delta\rho. \quad (27)
 \end{aligned}$$

Incorporating the contribution in free space and the nuclear medium effect given in Eqs. (24), (25), and (27), we have, for the in-medium η - π^0 mixing angle $\theta^{(\rho)}$,

$$\tan 2\theta^{(\rho)} = \frac{2}{m_\eta^2 - m_{\pi^0}^2} \left(\frac{m_1^2}{\sqrt{3}} + \left(\frac{g_A^2 m_\eta^2}{4\sqrt{3} f^2 m_N} + \frac{2c_5 m_\pi^2}{\sqrt{3} f^2} \right) \delta\rho + \frac{4c_1 m_1^2}{\sqrt{3} f^2} \rho \right), \quad (28)$$

where

$$m_1^2 = B_0 (m_d - m_u) = m_{K^0}^2 - m_{K^+}^2 - m_{\pi^0}^2 + m_{\pi^+}^2. \quad (29)$$

Equation (28) shows that the in-medium mixing angle depends not only on the asymmetric density $\delta\rho$ appearing in the second term, but also on the total baryon density ρ in the third term. We see that $\delta\rho$ enhances the mixing angle in the neutron-rich asymmetric nuclear medium, while ρ reduces the mixing angle because the coefficient c_1 is negative. The parameter c_1 is determined so as to reproduce the low-energy πN scattering [30] and the sign reflects the nature of the low-energy πN interaction. The resultant decay width is obtained from the balance of these effects. From the assignment of the isospin, the proton and neutron densities affect the decay in the same way as the u and d quark masses do. A larger density difference of the u and d quarks means a stronger violation of the isospin symmetry, so the η - π^0 mixing angle is enhanced in the neutron-rich nuclear medium.

A contour plot of the η - π^0 mixing angle in the asymmetric nuclear medium is presented in Fig. 2. In the present work, we use the following values in the numerical calculation [30]: $c_1 = -0.93 \pm 0.10 \text{ GeV}^{-1}$, $c_5 = -0.09 \pm 0.01 \text{ GeV}^{-1}$, and $f = 93 \text{ MeV}$. We note that the c_i of the LECs have uncertainties of some 10%. The masses of all the hadrons are taken to be the experimental values listed in Ref. [16] and thus $m_1^2 = 5165.86 \text{ MeV}^2$. One can find from this figure that the η - π^0 mixing angle is enhanced in the neutron-rich asymmetric nuclear medium, and tends to be slightly suppressed by the total baryon density.

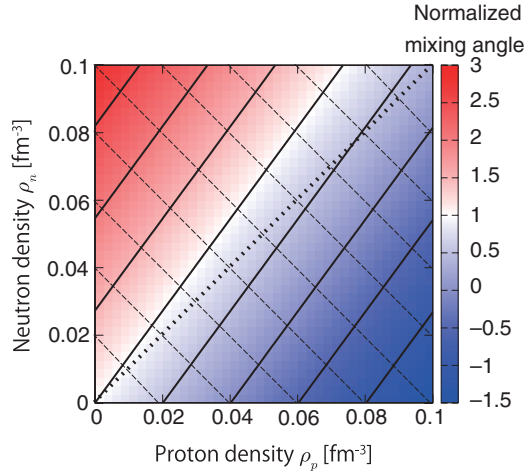


Fig. 2. The η - π^0 mixing angle in the asymmetric nuclear medium up to $O(q^5)$. The solid lines are the contour lines and are plotted per 0.5 of the mixing angle normalized with the free-space value. The horizontal and vertical axes are the proton density, ρ_p [fm^{-3}], and the neutron density, ρ_n [fm^{-3}]. The dashed and dotted lines are the constant- ρ and vanishing- $\delta\rho$ lines, respectively. The lower-left and upper-right regions of the figure correspond to small and large total baryon densities ρ , and the upper and lower sides of the dotted line are the neutron- and proton-rich regions, respectively. The value is normalized by the mixing angle in free space $\theta^{(0)} \simeq 1.058 \times 10^{-2}$ [rad.]. The mixing angle is smaller than that of the free-space value in the blue region where ρ_p is large, and larger in the red region where ρ_n is large.

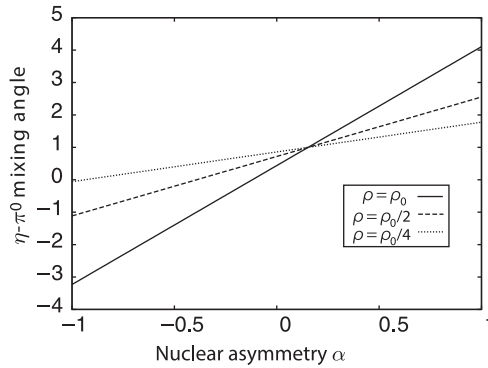


Fig. 3. The nuclear asymmetry α dependence of the η - π^0 mixing angle in the nuclear medium. The horizontal and vertical axes represent the nuclear asymmetry and the η - π^0 mixing angle normalized by the value at $\rho = \delta\rho = 0$. The solid, dashed, and dotted lines are the lines with $\rho = \rho_0$, $\rho_0/2$, and $\rho_0/4$, respectively.

We show the nuclear asymmetry α dependence of the η - π^0 mixing angle in Fig. 3: One sees that the mixing angle is enhanced by α , and the slope of the α dependence is bigger for the higher total baryon density. In fact, the slope of the mixing angle in terms of α in the small density is given as

$$\frac{d\theta}{d\alpha} \sim \frac{1}{2} \frac{d \tan 2\theta}{d\alpha} = \frac{\rho/\sqrt{3}f^2}{m_\eta^2 - m_{\pi^0}^2} \left\{ -4c_1 m_1^2 \alpha + \left(\frac{g_A^2 m_\eta^2}{4m_N} + 2c_5 m_\pi^2 \right) \right\}, \quad (30)$$

which is proportional to ρ .

The density dependence of the η - π^0 mixing angle has an uncertainty coming from those of the LECs, c_1 and c_5 [30]. The resultant uncertainty of the mixing angle is about 10%.

4. The η - 3π decay width in the asymmetric nuclear medium

In this section, we estimate the η - 3π decay width in the asymmetric nuclear medium. The partial width Γ in the rest frame of a particle with mass M reads

$$\Gamma = \frac{1}{n!} \frac{1}{4M} \int ds \int dt |\mathcal{M}|^2. \quad (31)$$

Here, n is the number of identical particles in the final state, \mathcal{M} the matrix element of the decay, and $s = (p_\eta - p_{\pi^0})^2$, $t = (p_\eta - p_{\pi^+})^2$ the Mandelstam variables.

The diagrams contributing to the η - 3π decay amplitude are shown in Fig. 4. Diagrams (a), (b), and (c) affect the decay amplitude through the η - π^0 mixing angle, and diagrams (d) to (g) give the medium effects on the decay amplitude directly. The in-medium η - π^0 mixing angle has already been calculated in Sect. 3, i.e., the contributions from diagrams (a), (b), and (c). We evaluate diagrams (d) to (g) in Fig. 4 and the decay width in the asymmetric nuclear medium in this section. In Sects. 4.1 and 4.2, we calculate the decay amplitude of η into $\pi^0\pi^+\pi^-$ and $3\pi^0$, respectively, and we show the results of the numerical estimation of the decay width in Sect. 4.3.

4.1. The η decay into $\pi^0\pi^+\pi^-$

In this subsection, we calculate the η decay amplitude into $\pi^0\pi^+\pi^-$ in the asymmetric nuclear medium. We show the matrix elements of η decay into $\pi^0\pi^+\pi^-$ that come from diagrams (d) to (g) in Fig. 4.

Diagrams (d) and (e) do not contribute to the decay amplitude because both the $\eta\pi^0\pi^+\pi^-\bar{N}N$ and the $\eta\pi^0\bar{N}N$ vertices coming from $\mathcal{L}_{\pi N}^{(1)}$ are equal to zero, as shown in Eq. (A14) and Appendix A.2.1.

With the use of the vertices of $\eta\bar{N}N$ and $\pi^0\pi^+\pi^-\bar{N}N$ given in Eqs. (A4) and (A13), the contribution from diagram (f) is written as

$$\begin{aligned} i\mathcal{M}^{(f)} = & (-) \left(-\frac{g_A}{2\sqrt{3}f} \right) \left(-\frac{g_A}{24f^3} \right) \int \frac{d^4p}{(2\pi)^4} \text{tr} \{ \gamma_5 \not{k} (\not{p} + \not{k} + m_N) \gamma_5 (2\not{p}_0 - \not{p}_+ - \not{p}_-) (\not{p} + m_N) \} \\ & \times \left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi\delta(p^2 - m_N^2) \theta(p_0) \theta(k_f - |\vec{p}|) \right\} \\ & \times \left\{ \frac{i}{(p+k)^2 - m_N^2 + i\epsilon} - 2\pi\delta((p+k)^2 - m_N^2) \theta(p_0 + k_0) \theta(k_f - |\vec{p} + \vec{k}|) \right\}. \quad (32) \end{aligned}$$

The minus sign comes from the fermion loop. Eliminating the pure-vacuum contributions and setting $A = 2p_0 - p_+ - p_-$, we have

$$\begin{aligned} i\mathcal{M}^{(f)} = & \frac{ig_A^2}{48\sqrt{3}f^4(2\pi)^3} \int d^4p \left\{ 2(k \cdot p)(A \cdot p) + k^2(A \cdot p) - p^2(A \cdot k) - m_N^2(A \cdot k) \right\} \quad (33) \\ & \times \left\{ \frac{1}{p^2 - m_N^2} \theta(p_0 + k_0) \delta((p+k)^2 - m_N^2) \theta(k_f - |\vec{p} + \vec{k}|) \right. \\ & \left. + \frac{1}{(p+k)^2 - m_N^2} \theta(p_0) \delta(p^2 - m_N^2) \theta(k_f - |\vec{p}|) \right\}. \quad (34) \end{aligned}$$

Changing the integration variable of the first term in the bracket p into $p' = p + k$ and keeping the leading order of the momentum, $i\mathcal{M}^{(f)}$ is reduced to

$$i\mathcal{M}^{(f)} = \frac{ig_A^2}{48\sqrt{3}f^4(2\pi)^3} \int d^3p \frac{2k^2(A \cdot k)}{2p \cdot k} \frac{1}{2E_N(\vec{p})} \times \theta(k_f - |\vec{p}|). \quad (35)$$

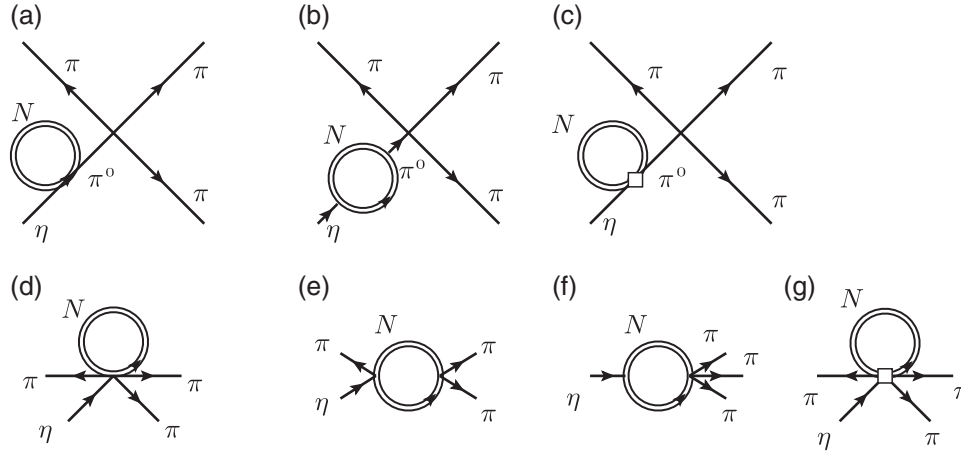


Fig. 4. The diagrams contributing to the η decay into three π . The meanings of the lines and vertices are same as in Fig. 1. Diagrams (a), (b), and (c) are the contributions from the mixing angle and (d) to (g) give the medium effects on the η - 3π decay amplitude directly.

Approximating $E_N(\vec{p})$ to m_N , we arrive at

$$i\mathcal{M}^{(f)} = i \frac{g_A^2}{48\sqrt{3}f^4} A_0 \rho. \quad (36)$$

Using the energy conservation, $A_0 = 3E_{\pi^0} - m_\eta$, and taking account of the opposite sign of the vertex between π and the proton or neutron, we find that the leading contribution containing the density effect finally has the following form:

$$i\mathcal{M}^{(f)} = i \frac{g_A^2}{48\sqrt{3}f^4} m_\eta \delta\rho - i \frac{g_A^2}{16\sqrt{3}f^4} E_{\pi^0} \delta\rho. \quad (37)$$

Here, we decompose the contribution from diagram (g) in Fig. 4 in two parts; the term proportional to c_1 and c_5 , respectively: $\mathcal{M}^{(g)} = \mathcal{M}^{(g1)} + \mathcal{M}^{(g5)}$, where $\mathcal{M}^{(g1)}$ and $\mathcal{M}^{(g5)}$ are proportional to c_1 and c_5 . The $\eta\pi^0\pi^+\pi^-\bar{N}N$ vertices proportional to c_1 and c_5 are presented in Eqs. (A21) and (A22), respectively.

With the $\eta\pi^0\pi^+\pi^-\bar{N}N$ vertex given in Eq. (A21), $\mathcal{M}^{(g1)}$ is given as

$$i\mathcal{M}^{(g1)} = - \left(-i \frac{4c_1 m_1^2}{3\sqrt{3}f^4} \right) \int \frac{d^4 p}{(2\pi)^4} \text{tr}(p + m_N) \times \left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi\delta(p^2 - m_N^2) \theta(p_0) \theta(k_f - |\vec{p}|) \right\}. \quad (38)$$

Eliminating the vacuum part of the nucleon propagator, $\mathcal{M}^{(g1)}$ is reduced to

$$i\mathcal{M}^{(g1)} = -i \frac{4c_1 m_1^2}{3\sqrt{3}f^4} \rho. \quad (39)$$

Using the $\eta 3\pi^0 \bar{N} N$ vertex given in Eq. (A22), $\mathcal{M}^{(g5)}$ is written as

$$\begin{aligned} i\mathcal{M}^{(g5)} = & - \left(i \frac{4c_5 m_u}{3\sqrt{3}} \right) \int \frac{d^4 p}{(2\pi)^4} \text{tr}(p + m_N) \\ & \times \left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi \delta(p^2 - m_N^2) \theta(p_0) \theta(k_f^{(p)} - |\vec{p}|) \right\} \\ & - \left(-i \frac{4c_5 m_d}{3\sqrt{3}} \right) \int \frac{d^4 p}{(2\pi)^4} \text{tr}(p + m_N) \\ & \times \left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi \delta(p^2 - m_N^2) \theta(p_0) \theta(k_f^{(n)} - |\vec{p}|) \right\} \\ & - \frac{1}{2} \left(i \frac{4c_5 m_1^2}{3\sqrt{3} f^4} \right) \int \frac{d^4 p}{(2\pi)^4} \text{tr}(p + m_N) \\ & \times \left[\left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi \theta(p_0) \delta(p^2 - m_N^2) \theta(k_f^{(p)} - |\vec{p}|) \right\} \right. \\ & \left. + \left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi \theta(p_0) \delta(p^2 - m_N^2) \theta(k_f^{(n)} - |\vec{p}|) \right\} \right]. \quad (40) \end{aligned}$$

Omitting the terms that come from the free-space propagation of the nucleon, we obtain $\mathcal{M}^{(g5)}$ as

$$i\mathcal{M}^{(g5)} = -i \frac{2c_5 m_\pi^2}{3\sqrt{3} f^4} \delta\rho. \quad (41)$$

Thus, $\mathcal{M}^{(g)}$ is written as

$$i\mathcal{M}^{(g)} = -i \frac{4c_1 m_1^2}{3\sqrt{3} f^4} \rho - i \frac{2c_5 m_\pi^2}{3\sqrt{3} f^4} \delta\rho. \quad (42)$$

Taking account of the contributions from free space and the modification of the η - π^0 mixing angle, we have the matrix element of the η decay into $\pi^0 \pi^+ \pi^-$ in the asymmetric nuclear medium up to $O(q^5)$ as

$$\begin{aligned} \mathcal{M}_{\eta \rightarrow \pi^0 \pi^+ \pi^-} = & - \left(\frac{m_1^2}{3\sqrt{3} f^2} + \frac{s - s_0}{f^2} \sin \theta^{(\rho)} \right) + \mathcal{M}_{\eta \rightarrow \pi^0 \pi^+ \pi^-}^{(4)} \\ = & - \frac{m_1^2}{3\sqrt{3} f^2} \left(1 + \frac{3(s - s_0)}{m_\eta^2 - m_{\pi^0}^2} \right) + \sin \theta^{(0)} \mathcal{M}_{\eta \rightarrow \pi^0 \pi^+ \pi^-}^{(4)\text{vac}} \\ & + \left\{ - \frac{s - s_0}{f^2} \frac{1}{m_\eta^2 - m_{\pi^0}^2} \left(\frac{g_A^2 m_\eta^2}{4\sqrt{3} f^2} + \frac{2c_5 m_\pi^2}{\sqrt{3} f^2} \right) + \frac{g_A^2}{48\sqrt{3} f^4} (m_\eta - E_{\pi^0}) - \frac{2c_5 m_\pi^2}{3\sqrt{3} f^4} \right\} \delta\rho \\ & - \frac{4c_1 \rho}{f^2} \frac{m_1^2}{3\sqrt{3} f^2} \left(1 + \frac{3(s - s_0)}{m_\eta^2 - m_{\pi^0}^2} \right), \quad (43) \end{aligned}$$

where

$$\mathcal{M}_{\eta \rightarrow \pi^0 \pi^+ \pi^-}^{(4)} = \sin \theta^{(0)} \mathcal{M}_{\eta \rightarrow \pi^0 \pi^+ \pi^-}^{(4)\text{vac}} + \mathcal{M}^{(f)} + \mathcal{M}^{(g)}. \quad (44)$$

Here, s_0 is given as $s_0 = m_\eta^2/3 + m_\pi^2$, and $\mathcal{M}_{\eta \rightarrow \pi^0 \pi^+ \pi^-}^{(4)\text{vac}}$ is the meson one-loop contribution in free space, which is known to give a large contribution, as mentioned in Sect. 1. The details of the form

and calculation of $\mathcal{M}_{\eta \rightarrow \pi^0 \pi^+ \pi^-}^{(4)\text{vac}}$ are given in Refs. [5,7]. $\mathcal{M}^{(f)}$ and $\mathcal{M}^{(g)}$ are given in Eqs. (37) and (42), respectively. We denote the η - π^0 mixing angle in free space by $\theta^{(0)}$, which is given by setting $\rho = \delta\rho = 0$ in Eq. (28). In this calculation, we have assumed that the isospin symmetry breaking is so small that we can make the approximation that $\sin \theta \sim \theta \sim \tan 2\theta/2$.

4.2. The η decay into three π^0

In this subsection, we give the decay amplitude for the η decay to three π^0 in the asymmetric nuclear medium.

First of all, the contributions from diagrams (d), (e), and (f) vanish in the three- π^0 case. Diagrams (d) and (e) give no contribution in the same way as in the case of the $\pi^0 \pi^+ \pi^-$ decay. The contribution of diagram (f) also vanishes because the $3\pi^0 \bar{N}N$ vertex is zero, as is shown in Appendix A3. Thus, the decay amplitude in the asymmetric nuclear medium is solely given by the sum of the contributions from the diagram (g). $\mathcal{M}^{(g1)}$, which is proportional to c_1 , is written as

$$i\mathcal{M}^{(g1)} = - \left(-i \frac{2c_1 m_1^2}{3\sqrt{3}f^4} \right) \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi\delta(p^2 - m_N^2)\theta(p_0)\theta(k_f - |\vec{p}|) \right\}, \quad (45)$$

where the $\eta 3\pi^0 \bar{N}N$ vertex appearing from c_1 is presented in Eq. (A18). The nuclear medium modification is evaluated to be

$$i\mathcal{M}^{(g1)} = -i \frac{2m_1^2 c_1}{3\sqrt{3}f^4} \rho. \quad (46)$$

The $\eta 3\pi^0 \bar{N}N$ vertex with c_5 is given in Eq. (A22), and the term proportional to c_5 in $\mathcal{M}^{(g5)}$ reads

$$\begin{aligned} i\mathcal{M}^{(g5)} = & (-i) \frac{2c_5 B_0}{3\sqrt{3}f^4} \int \frac{d^4 p}{(2\pi)^4} \text{tr}(\not{p} + m_N) \\ & \times \left(m_u \left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi\delta(p^2 - m_N^2)\theta(p_0)\theta(k_f^{(p)} - |\vec{p}|) \right\} \right. \\ & \left. - m_d \left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi\delta(p^2 - m_N^2)\theta(p_0)\theta(k_f^{(n)} - |\vec{p}|) \right\} \right) \\ & - \frac{1}{2} \left(i \frac{2c_5 m_1^2}{3\sqrt{3}f^4} \right) \int \frac{d^4 p}{(2\pi)^4} \text{tr}(\not{p} + m_N) \\ & \times \left(\left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi\delta(p^2 - m_N^2)\theta(p_0)\theta(k_f^{(p)} - |\vec{p}|) \right\} \right. \\ & \left. + \left\{ \frac{i}{p^2 - m_N^2 + i\epsilon} - 2\pi\delta(p^2 - m_N^2)\theta(p_0)\theta(k_f^{(n)} - |\vec{p}|) \right\} \right). \quad (47) \end{aligned}$$

Omitting the free-space part in the right-hand side of the equation, we obtain

$$i\mathcal{M}^{(g5)} = -i \frac{c_5 m_{\pi^0}^2}{3\sqrt{3}f^4} \delta\rho. \quad (48)$$

Summing up Eqs. (46) and (48), we have

$$i\mathcal{M}^{(g)} = -i \frac{2m_1^2 c_1}{3\sqrt{3}} \rho - i \frac{c_5 m_{\pi^0}^2}{3\sqrt{3}f^4} \delta\rho. \quad (49)$$

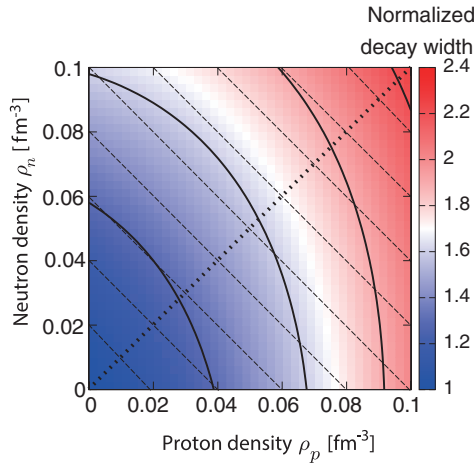


Fig. 5. $\eta \rightarrow \pi^0 \pi^+ \pi^-$ decay width in asymmetric nuclear medium up to $O(q^5)$. The horizontal and vertical axes represent ρ_p and ρ_n , respectively. The solid line represents the contour of the decay width. The dashed and dotted lines mean the constant- ρ and the vanishing- $\delta\rho$ lines, respectively. The width is normalized by the value at $\rho = \delta\rho = 0$. The contour is plotted per 0.3. The width in free space is 163 eV.

Thus we have the matrix element of the η decay into $3\pi^0$ as

$$\begin{aligned} \mathcal{M}_{\eta \rightarrow 3\pi^0} &= -\frac{m_1^2}{\sqrt{3}f^2} + \mathcal{M}_{\eta \rightarrow 3\pi^0}^{(4)} \\ &= -\frac{m_1^2}{\sqrt{3}f^2} + \sin\theta^{(0)} \mathcal{M}_{\eta \rightarrow 3\pi^0}^{(4)\text{vac}} - \frac{2m_1^2 c_1}{3\sqrt{3}f^4} \rho - \frac{c_5 m_\pi^2}{3\sqrt{3}f^4} \delta\rho, \end{aligned} \quad (50)$$

with

$$\mathcal{M}_{\eta \rightarrow 3\pi^0}^{(4)} = \sin\theta^{(0)} \mathcal{M}_{\eta \rightarrow 3\pi^0}^{(4)\text{vac}} + \mathcal{M}^{(g)}. \quad (51)$$

Here, $\mathcal{M}_{\eta \rightarrow 3\pi^0}^{(4)\text{vac}}$ is the contribution from the meson one-loop and its detailed form is presented in Refs. [5,7]. We note that the amplitude does not depend on the η - π^0 mixing angle in the leading order, on account of the symmetry of the final state consisting of three identical π^0 . For this reason, the medium modification of the η decay into three π^0 is small, which will be demonstrated in the numerical calculation given in the next subsection.

4.3. Numerical results

Using the definition of the decay width given in Eq. (31) and the matrix elements presented in Eqs. (43) and (50), we evaluate the partial width of the η decay into 3π .

Figures 5 and 6 are contour plots of the decay width of η into $\pi^0 \pi^+ \pi^-$ and into three π^0 , respectively. The widths are normalized with each value at $\rho = \delta\rho = 0$, respectively.

First, we discuss the η - $\pi^0 \pi^+ \pi^-$ decay width. From Fig. 5, one finds that the width is large in the higher-density region (upper right of the figure), and the decay width is enhanced in the proton-rich region. We show the $\delta\rho$ dependence of the decay width with some values of fixed ρ in Fig. 7.

The parabolic-shape dependence on $\delta\rho$ comes from the $\sin^2\theta$ dependence of the decay width: The minimum points of the decay width are not located at $\alpha = 0$ due to the explicit symmetry breaking in free space caused by the different u, d quark masses. We find that the decay width is enhanced by the total baryon density ρ .

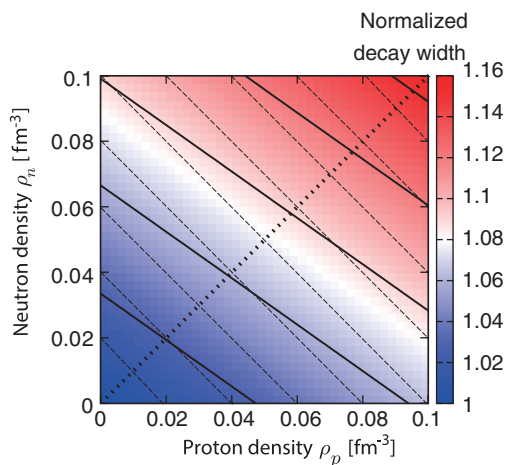


Fig. 6. $\eta \rightarrow 3\pi^0$ decay width in the asymmetric nuclear medium up to $O(q^5)$. The axes and the lines are the same as those in Fig. 5. The contour is plotted per 0.03. The width in free space is 298 eV.

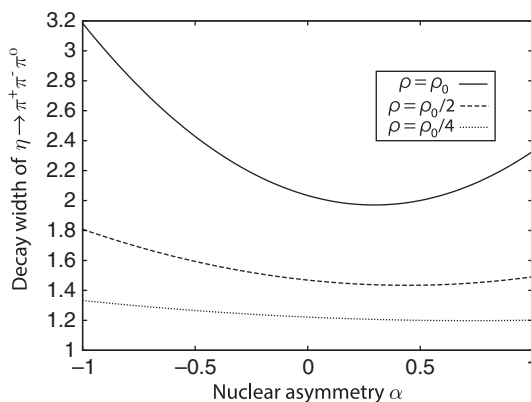


Fig. 7. The nuclear asymmetry α dependence of the η decay width into $\pi^0\pi^+\pi^-$ with some fixed ρ . The solid, dashed, and dotted lines are the decay widths at $\rho = \rho_0$, $\rho_0/2$, and $\rho_0/4$, respectively. The decay width is normalized with the value at ρ and $\delta\rho = 0$.

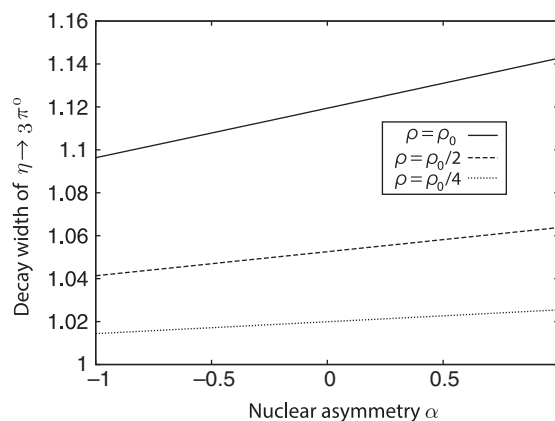


Fig. 8. The nuclear asymmetry α dependence of the η decay width into three π^0 with some fixed ρ . The solid, dashed, and dotted lines are the decay widths at $\rho = \rho_0$, $\rho_0/2$, and $\rho_0/4$, respectively. The decay width is normalized with the value at ρ and $\delta\rho = 0$.

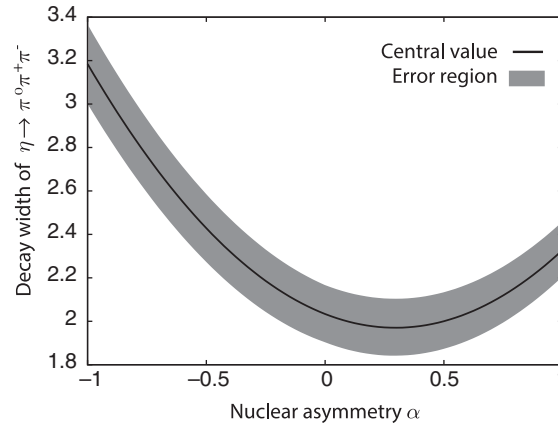


Fig. 9. The uncertainty of the η decay width into $\pi^0\pi^+\pi^-$ at normal nuclear density. The solid line corresponds to the decay width with the LECs at the central values. The shaded area represents the uncertainty of the width due to that of the LECs. The others are the same as those of Fig. 7.

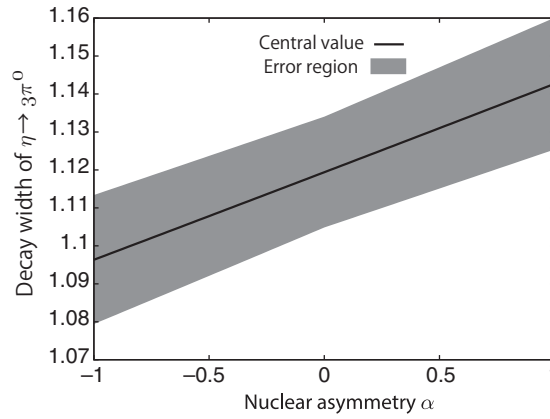


Fig. 10. The uncertainty of the η decay width into $3\pi^0$ due to that of the LECs at normal nuclear density. The line and the shaded area are same as those of Fig. 9.

Next, we discuss the η - $3\pi^0$ decay. Figure 6 is a contour plot of the η - $3\pi^0$ decay in the asymmetric nuclear medium and Fig. 8 shows the $\delta\rho$ dependence of the η - $3\pi^0$ decay width with some fixed values of the total density ρ . The decay width of η into $3\pi^0$ shows an enhancement by the total baryon density in much the same way as η into $\pi^0\pi^+\pi^-$ decay. This enhancement is caused by the third term in Eq. (50). One finds that the effect of the total baryon density ρ overwhelms that of the isospin asymmetry $\delta\rho$ on the η - $3\pi^0$ decay. Nonetheless, the neutron-rich medium enhances the decay width, as one can see in Fig. 8. The relative smallness of the effect of the isospin asymmetry $\delta\rho$ on the decay in comparison with that on the $\pi^0\pi^+\pi^-$ decay can be attributed to the fact that the mixing angle dependence of the decay amplitude is suppressed by the crossing symmetry.

Here, we comment on the uncertainties of the decay widths coming from those of the LECs. The uncertainties of the η decay width into $\pi^0\pi^+\pi^-$ and $3\pi^0$ are both about 10%. We present Figs. 9 and 10 to visualize the uncertainties for normal nuclear densities as an example: The solid lines are the central values and the shaded areas show the uncertainties from the LECs.

Here, we discuss the origin of the enhancement of the decay widths with the total baryon density. It is found that the dominant density dependence for both decays comes from the term that is proportional to the low-energy constant c_1 in Eqs. (43) and (50) for the $\pi^0\pi^+\pi^-$ and $3\pi^0$ decay,

respectively. The parameter c_1 is related to $\sigma_{\pi N}$ by $c_1 = -\frac{\sigma_{\pi N}}{4m_\pi^2}$ because we assume that all the loop corrections of the nucleon mass are renormalized into LECs [32]. Substituting this relation into the second term of Eq. (43), one finds that the prefactor of the second term reads

$$-\frac{4c_1 m_1^2}{3\sqrt{3}f^4}\rho = \frac{\sigma_{\pi N}\rho}{m_\pi^2 f^2} \frac{m_1^2}{3\sqrt{3}f^2}.$$

It is noteworthy here that the coefficient $\sigma_{\pi N}\rho/m_\pi^2 f^2$ is nothing but the quantity that represents the reduction rate of the quark condensate or chiral order parameter in the nuclear medium with the linear density approximation [33,34];

$$\frac{\delta \langle \bar{q}q \rangle}{\langle \bar{q}q \rangle_{\rho=0}} \equiv \frac{\langle \bar{q}q \rangle_{\rho=0} - \langle \bar{q}q \rangle_\rho}{\langle \bar{q}q \rangle_{\rho=0}} = \frac{\sigma_{\pi N}}{m_\pi^2 f^2} \rho, \quad (52)$$

where $\langle \bar{q}q \rangle_{\rho=0}$ and $\langle \bar{q}q \rangle_\rho$ are the quark condensates at $\rho = 0$ and non-zero, respectively. Thus, one should be able to rewrite the decay amplitude given in Eq. (43) (Eq. (50)) for the $\pi^0\pi^+\pi^-$ ($3\pi^0$) decay in terms of the reduction of the chiral order parameter $\delta \langle \bar{q}q \rangle$. Here, we rewrite the $\mathcal{M}_{\eta \rightarrow \pi^0\pi^+\pi^-}$ presented in Eq. (43) in terms of the renormalized pion decay constant f^* :

$$\begin{aligned} \mathcal{M}_{\eta \rightarrow \pi^0\pi^+\pi^-} &= -\frac{m_1^2}{3\sqrt{3}f^2} \left(1 + \frac{\sigma_{\pi N}}{m_\pi^2 f^2} \rho \right) \left(1 + \frac{3(s-s_0)}{m_\eta^2 - m_{\pi^0}^2} \right) + \sin \theta^{(0)} \mathcal{M}_{\eta \rightarrow \pi^0\pi^+\pi^-}^{(4)\text{vac}} \\ &\quad + \left\{ -\frac{s-s_0}{f^2} \frac{1}{m_\eta^2 - m_{\pi^0}^2} \left(\frac{g_A^2 m_\eta^2}{4\sqrt{3}f^2} + \frac{2c_5 m_\pi^2}{\sqrt{3}f^2} \right) + \frac{g_A^2}{48\sqrt{3}f^4} (m_\eta - E_{\pi^0}) - \frac{2c_5 m_\pi^2}{3\sqrt{3}f^4} \right\} \delta\rho \\ &= -\frac{m_1^2}{3\sqrt{3}f^{*2}} \left(1 + \frac{3(s-s_0)}{m_\eta^2 - m_{\pi^0}^2} \right) + \sin \theta^{(0)} \mathcal{M}_{\eta \rightarrow \pi^0\pi^+\pi^-}^{(4)\text{vac}} \\ &\quad + \left\{ -\frac{s-s_0}{f^2} \frac{1}{m_\eta^2 - m_{\pi^0}^2} \left(\frac{g_A^2 m_\eta^2}{4\sqrt{3}f^2} + \frac{2c_5 m_\pi^2}{\sqrt{3}f^2} \right) + \frac{g_A^2}{48\sqrt{3}f^4} (m_\eta - E_{\pi^0}) - \frac{2c_5 m_\pi^2}{3\sqrt{3}f^4} \right\} \delta\rho, \end{aligned} \quad (53)$$

where $f^{*2} = f^2 \left(1 - \frac{\sigma_{\pi N}}{f^2 m_\pi^2} \rho \right)$. From the first to the second line, we have regarded $\sigma_{\pi N}\rho/m_\pi^2 f^2$ as a small quantity, and thus the ρ dependence is absorbed into f^{*2} . As one can see in Eq. (53), the total baryon density dependence of the decay amplitude of the $\eta \rightarrow \pi^0\pi^+\pi^-$ process can be renormalized into the density dependence of the pion decay constant⁴. Thus, one may say that the enhancement of the 3π decay width originates from the chiral restoration in the nuclear medium, although more detailed analysis is necessary to establish the relevance of the partial restoration of the chiral symmetry in the enhancement of the 3π decay of η at finite baryon density. Nevertheless, it is worth mentioning that a similar mechanism has been identified as being responsible for an enhancement of the $\pi\pi$ cross section near the 2π threshold in the σ meson channel in nuclear matter

⁴ We note that Eq. (52) can also be rewritten in terms of f^{*2} under the assumption of the smallness of the change in the pion mass in the nuclear medium and the in-medium Gell-Mann–Oakes–Renner relation [35]:

$$\frac{f_\pi^{*2} m_\pi^{*2}}{f_\pi^2 m_\pi^2} = \frac{\langle \bar{q}q \rangle_\rho}{\langle \bar{q}q \rangle_{\rho=0}}. \quad (54)$$

by Jido, Hatsuda, and one of the present authors [36], where it is found that the reduction of the chiral condensate implying the partial restoration of chiral symmetry in nuclear matter is responsible for this enhancement. Furthermore, they clarified that a 4π -nucleon vertex shown in Fig. 5 in Ref. [36], which has the same structure as diagram (g) in Fig. 4 in the present article, is responsible for the enhancement.

5. Brief summary and concluding remarks

In this paper, we have studied the η - π^0 mixing angle defined in Eq. (1) and the η decay into 3π in the nuclear medium with varying isospin asymmetry on the basis of the in-medium chiral effective theory, where the Fermi momentum k_f as well as the pseudoscalar meson masses and momenta are treated as small expansion parameters. We have found that both the quantities are significantly modified in the nuclear medium by the asymmetry $\delta\rho = \rho_n - \rho_p$ and the total baryon densities $\rho = \rho_n + \rho_p$, although the densities affect both the quantities in different manners: the mixing angle increases along with the asymmetric density $\delta\rho$, but decreases along with the increase of the total baryon density. In terms of the α dependence of the mixing angle, the mixing angle increases along with α , the slope of which is greater for larger total baryon density. The increase of the mixing angle means that the physical η meson has a greater π_3 component, the isospin eigenstate of the neutral pion. It turns out that the total and asymmetric densities tend to increase the decay width of η into $\pi^0\pi^+\pi^-$ in an additive way, although the total density dependence overwhelms that of the isospin. For example, the width is enhanced by a factor of 2 to 3 at $\rho = \rho_0$ depending on the isospin asymmetry α .

The enhancement of the width due to the isospin asymmetry is traced back to the increase of the π_3 component in the physical η state.

The η - $3\pi^0$ decay width in the nuclear medium is enhanced by the total baryon density ρ as in the case of $\eta \rightarrow \pi^0\pi^+\pi^-$ decay, although the enhancement is relatively smaller than the latter case. This is because the mixing angle dependence of the η - $3\pi^0$ decay amplitude is suppressed by symmetrization of the three π^0 in the final state. The neutron-rich asymmetric medium slightly enhances the decay width in the η - $3\pi^0$ case.

The ρ -dependent parts of the decay amplitudes of the η to 3π decay given in Eqs. (43) and (50) are both proportional to the low-energy constant c_1 , which is in turn related to the π - N sigma term $\sigma_{\pi N}$. Thus, we find that the effect of the total baryon density can be rewritten in terms of the change of the quark condensate in the nuclear medium. This suggests that the enhancement of the decay rate may result from the partial restoration of chiral symmetry in the nuclear medium. Further analyses are, however, necessary to elucidate the underlying mechanism of the enhancement and its possible relationship with the chiral restoration in the nuclear medium.

It may be possible to observe the modification with the nucleus target experiment with large isospin asymmetry $(N - Z)/A$ at facilities such as, e.g., SPring-8 and J-PARC in Japan, or FAIR, COSY, and MAMI in Germany.

In our calculation, the decay widths in free space reproduce the experimental data fairly well: The values are roughly 60% in the $\eta \rightarrow \pi^0\pi^+\pi^-$ case and 70% in the $\eta \rightarrow 3\pi^0$ case compared with the experimental data [16]. To reproduce the experimental data in free space more precisely, the higher-order terms in the chiral perturbation should be included [7]. Some of them may be resummed into the final-state interaction of π , more precisely, the s-wave $\pi^+\pi^-$ correlations, which may cause the σ resonance, as mentioned in Sect. 1.

The nuclear medium would affect the spectral properties of the σ mode through chiral restoration [36–39] (see also, e.g., Refs. [17,18]), and thus a full account of the medium effect on the σ mode including its possible softening on may have a significant impact on the η decay in the nuclear medium.

Our calculations do not take the effect of the $N^*(1535)$ resonance into account, as mentioned in Sect. 3. It is known [31] (see also, e.g., Ref. [24] for a recent review) that the coupling with the resonance modifies the in-medium self-energy or the optical potential of η : It is suggested that the η – N interaction is attractive, and the attraction might lead to a reduction of the η mass, say of 50 MeV order, in the nuclear medium. Furthermore, there arises an induced η – π^0 coupling through the N^* –nucleon hole excitations in the nuclear medium. The formula of the η – π^0 mixing angle in Eq. (1) tells us that both the reduction of the η mass and the additional η – π^0 coupling would cause an additional ρ dependence but not the $\delta\rho$ dependence of the mixing angle and hence an enhancement of the $\pi^0\pi^+\pi^-$ decay width. On the other hand, the $3\pi^0$ decay is independent of the mixing angle, as shown in Eq. (50), so the effect of the resonance on the $3\pi^0$ decay width should be small. We hope that we can report on a quantitative analysis of such an additional density dependence coming from the coupling with N^* of the mixing angle and the decay width of η in the near future.

Recently, the possible effects of an external magnetic field on hadron properties have been the focus of intensive studies; the relevant physical quantities include mass spectra of light hadrons [40,41] and pseudoscalar–vector mixing rates in heavy quarkonia, as well as their mass shifts [42,43]. We note that a strong magnetic field could also be a possible source of isospin symmetry breaking due to the difference in the electromagnetic charges of the u and d quarks, and thus an external magnetic field may change the hadron properties, as we have shown in the present work. This subject is left for future study.

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Appendix A. Meson–baryon vertices

In this appendix, we present the meson–baryon vertices used in the calculation in the text.

We first recall the following formula for the derivative of the exponential operator:

$$e^{-iA(t)} \frac{d}{dt} e^{iA(t)} = i \frac{dA}{dt} + \frac{1}{2!} \left[A, \frac{dA}{dt} \right] - \frac{i}{3!} \left[A, \left[A, \frac{dA}{dt} \right] \right] - \frac{1}{4!} \left[A, \left[A, \left[A, \frac{dA}{dt} \right] \right] \right] + \cdots \quad (\text{A1})$$

In the Lagrangian (Eq. (2)), $u^\dagger \partial_\mu u$ and $u \partial_\mu u^\dagger$ appear. With the definition of $u = e^{i\pi/2f}$, $u^\dagger \partial_\mu u$ can be expanded as

$$\begin{aligned} u^\dagger \partial_\mu u &= e^{-i\pi/2f} \partial_\mu e^{i\pi/2f} = \frac{i}{2f} \partial_\mu \pi + \frac{1}{2!(2f)^2} [\pi, \partial_\mu \pi] - \frac{i}{3!(2f)^3} [\pi, [\pi, \partial_\mu \pi]] \\ &\quad - \frac{1}{4!(2f)^4} [\pi, [\pi, [\pi, \partial_\mu \pi]]] + \cdots \end{aligned} \quad (\text{A2})$$

In the case of $u \partial_\mu u^\dagger$, the terms with odd powers of the π field change their signs.

A.1. $\eta\bar{N}N$ and $\pi^0\bar{N}N$ vertices

With the definitions of Γ_μ and u , the $\eta\bar{N}N$ and $\pi^0\bar{N}N$ vertices from $\mathcal{L}_{\pi N}^{(1)}$ (Eq. (5)) are given as

$$i\bar{N}\frac{g_A}{2}\gamma^\mu\gamma_5i\partial_\mu\left(\frac{i\pi}{2f}-\frac{-i\pi}{2f}\right)N=\bar{N}\left(\frac{ig_A}{2f}\gamma_5\gamma^\mu\partial_\mu\pi\right)N, \quad (\text{A3})$$

so the $\eta\bar{N}N$ and $\pi^0\bar{N}N$ vertices, $g_{\eta\bar{N}N}$ and $g_{\pi^0\bar{N}N}$, are given as

$$g_{\eta\bar{N}N}=-\frac{g_A}{2f\sqrt{3}}\gamma_5\mathbf{k}\mathbf{1} \quad (\text{A4})$$

$$g_{\pi^0\bar{N}N}=-\frac{g_A}{2f}\gamma_5\mathbf{k}\tau_3. \quad (\text{A5})$$

Here, the 2×2 matrices τ_3 and $\mathbf{1}$ operate on the nucleon doublet field.

A.2. $\eta\pi^0\bar{N}N$ vertex

A.2.1. The vertex from $\mathcal{L}_{\pi N}^{(1)}$

The $\eta\pi^0\bar{N}N$ vertex in the leading order comes from $\mathcal{L}_{\pi N}^{(1)}$ presented in Eq. (5). The quadratic terms are induced from the covariant derivative defined by Eq. (12). Using Eq. (A2) and leaving the quadratic term of π , we obtain the $\pi^a\pi^b\bar{N}N$ vertex as

$$i\bar{N}\gamma^\mu\frac{i}{2}\left(u^\dagger\partial_\mu u+u\partial_\mu u^\dagger\right)N=i\bar{N}\gamma^\mu\frac{i}{2!(2f)^2}[\pi,\partial_\mu\pi]N. \quad (\text{A6})$$

In the $\eta\pi^0\bar{N}N$ case, note that λ^3 and λ^8 commute. Thus, we see that the $\eta\pi^0\bar{N}N$ vertex from $\mathcal{L}_{\pi N}^{(1)}$ vanishes.

A.2.2. The vertex from the c_1 term

The $\eta\pi^0\bar{N}N$ vertex with c_1 is given as

$$\begin{aligned} & ic_1\left\langle u^\dagger\chi u^\dagger+u\chi u\right\rangle\bar{N}N \\ &= ic_1\left\langle\left(\chi\frac{1}{2!}\left(\frac{-i\pi}{2f}\right)^2+\left(\frac{-i\pi}{2f}\right)\chi\left(\frac{-i\pi}{2f}\right)+\frac{1}{2!}\left(\frac{-i\pi}{2f}\right)^2\chi\right)\right. \\ &\quad \left.+\left(\chi^\dagger\frac{1}{2!}\left(\frac{i\pi}{2f}\right)^2+\left(\frac{i\pi}{2f}\right)\chi^\dagger\left(\frac{i\pi}{2f}\right)+\frac{1}{2!}\left(\frac{i\pi}{2f}\right)^2\chi^\dagger\right)\right\rangle\bar{N}N \\ &= -i\frac{8B_0c_1}{4f^2}\left\langle\mathcal{M}\pi^2\right\rangle\bar{N}N \\ &= -\frac{2B_0c_1}{f^2}\left\langle\begin{pmatrix}m_u & \\ & m_d\end{pmatrix}\begin{pmatrix}\pi^0+\eta/\sqrt{3} & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi^0+\eta/\sqrt{3}\end{pmatrix}\right\rangle\bar{N}N \\ &= -i\frac{2B_0c_1}{f^2}\left\langle\begin{pmatrix}m_u & \\ & m_d\end{pmatrix}\begin{pmatrix}\left(\pi^0+\eta/\sqrt{3}\right)^2+2\pi^+\pi^- & 2\sqrt{2/3}\pi^-\eta \\ 2\sqrt{2/3}\pi^+\eta & \left(-\pi^0+\eta/\sqrt{3}\right)^2+2\pi^+\pi^- \end{pmatrix}\right\rangle\bar{N}N. \end{aligned} \quad (\text{A7})$$

Thus, the $\eta\pi^0\bar{N}N$ vertex from the c_1 term is given as

$$g_{\eta\pi^0\bar{N}N}^{(c_1)}=-i\frac{4B_0c_1}{\sqrt{3}f^2}(m_u-m_d)=i\frac{4c_1m_1^2}{\sqrt{3}f^2}. \quad (\text{A8})$$

A.2.3. The vertex from the c_5 term

The $\eta\pi^0\bar{N}N$ vertex from the c_5 vertex is obtained as

$$\begin{aligned} i c_5 \left\langle \bar{N} \left(\chi \frac{1}{2!} \left(\frac{-i\pi}{2f} \right)^2 + \left(\frac{-i\pi}{2f} \right) \chi \left(\frac{-i\pi}{2f} \right) + \frac{1}{2!} \left(\frac{-i\pi}{2f} \right)^2 \chi \right) N \right. \\ \left. + \bar{N} \left(\chi^\dagger \frac{1}{2!} \left(\frac{i\pi}{2f} \right)^2 + \left(\frac{i\pi}{2f} \right) \chi^\dagger \left(\frac{i\pi}{2f} \right) + \frac{1}{2!} \left(\frac{i\pi}{2f} \right)^2 \chi^\dagger \right) N \right\rangle - i \frac{c_5}{2} \frac{4m_1^2}{\sqrt{3}f^2} \bar{N}N \\ = -i \frac{2B_0c_5}{8f^2} \left\langle \bar{N} \left(\mathcal{M}\pi^2 + 2\pi\mathcal{M}\pi + \pi^2\mathcal{M} \right) N \right\rangle - i \frac{c_5}{2} \frac{4m_1^2}{\sqrt{3}f^2} \bar{N}N. \end{aligned} \quad (\text{A9})$$

The relevant terms to $\eta\pi^0\bar{N}N$ are

$$\begin{aligned} = -i \frac{2B_0c_5}{f^2} \bar{N} \begin{pmatrix} m_u & \\ & m_d \end{pmatrix} \begin{pmatrix} 2\pi^0\eta/\sqrt{3} & \\ & -2\pi^0\eta/\sqrt{3} \end{pmatrix} N - i \frac{c_5}{2} \frac{4m_1^2}{\sqrt{3}f^2} \bar{N}N \\ = -i \frac{2B_0c_5}{f^2} \left((\bar{p}p)m_u(2\pi^0\eta/\sqrt{3}) - (\bar{n}n)m_d(2\pi^0\eta/\sqrt{3}) \right) - i \frac{2c_5m_1^2}{\sqrt{3}f^2} (\bar{p}p + \bar{n}n). \end{aligned} \quad (\text{A10})$$

Thus, the $\eta\pi^0\bar{N}N$ vertex from the c_5 term is given by

$$g_{\eta\pi^0\bar{N}N}^{(c_5)} = -i \frac{4B_0c_5}{\sqrt{3}f^2} \text{diag}(m_u, -m_d) - i \frac{2c_5m_1^2}{\sqrt{3}} \mathbf{1}. \quad (\text{A11})$$

A.3. $\pi^0\pi^+\pi^-\bar{N}N$ vertex

The $\pi^0\pi^+\pi^-\bar{N}N$ vertex from $\mathcal{L}_{\pi N}^{(1)}$ is given as

$$\begin{aligned} i\bar{N} \left(\frac{g_A}{2} \gamma^\mu \gamma_5 i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger) \right) = i\bar{N} \left(\frac{g_A}{2} \gamma^\mu \gamma_5 (2i) \left(-\frac{i}{3!(2f)^3} \right) [\pi, [\pi, \partial_\mu \pi]] \right) N \\ = i\bar{N} \left(\frac{g_A}{2} \gamma^\mu \gamma_5 (2i) \left(-\frac{i}{3!(2f)^3} \right) [\lambda^a, [\lambda^b, \lambda^c]] \pi^a \pi^b \partial_\mu \pi^c \right) N \\ = \frac{-g_A}{48f^3} \bar{N} \gamma^\mu \pi^a \pi^b \partial_\mu \pi^c [\lambda^a, [\lambda^b, \lambda^c]] N. \end{aligned} \quad (\text{A12})$$

If the η field is contained in the π , the commutator vanishes because the η field commutes with all the $\pi^{0,\pm}$ fields. With the Fourier transformation and the formula $[\lambda^1 + \lambda^2, [\lambda^1 + \lambda^2, \lambda^3]] = 2\lambda^3$, the $\pi^0\pi^+\pi^-\bar{N}N$ vertex is obtained as

$$g_{\pi^0\pi^+\pi^-\bar{N}N} = -\frac{1}{24f^3} \gamma_5 \gamma_\mu (2p_0^\mu - p_+^\mu - p_-^\mu). \quad (\text{A13})$$

Because $a, b, c = \pi^0$ in Eq. (A12), the $3\pi^0$ vertex vanishes.

A.4. $\eta\pi^0\pi^+\pi^-\bar{N}N$ and $\eta 3\pi^0\bar{N}N$ vertices

A.4.1. The vertex from $\mathcal{L}_{\pi N}^{(1)}$

The four mesons and the $\bar{N}N$ vertex from $\mathcal{L}_{\pi N}^{(1)}$ are given as

$$\begin{aligned} i\bar{N} \gamma^\mu \frac{i}{2} (u \partial_\mu u^\dagger + u^\dagger \partial_\mu u) N = \bar{N} \left(\frac{1}{4!(2f)^4} [\pi, [\pi, [\pi, \partial_\mu \pi]]] \right) N \\ = \bar{N} \left(\frac{1}{4!(2f)^4} [\lambda^a, [\lambda^b, [\lambda^c, \lambda^d]]] \pi^a \pi^b \pi^c \partial_\mu \pi^d \right) N. \end{aligned} \quad (\text{A14})$$

Here, we consider the quartic terms of π of Eq. (A2) because they are relevant to the $\eta\pi^0\pi^+\pi^-\bar{N}N$ and $\eta3\pi^0\bar{N}N$ vertices. Because the η field is contained, the commutator of the Gell-Mann matrices vanishes. Accordingly, the $\eta\pi^0\pi^+\pi^-\bar{N}N$ and $\eta3\pi^0\bar{N}N$ vertices in $\mathcal{L}_{\pi N}^{(1)}$ vanish.

A.4.2. The vertex from the c_1 term

The $\eta\pi^0\pi^+\pi^-\bar{N}N$ vertex from the c_1 term is given as

$$\begin{aligned} & i c_1 \left(u^\dagger \chi u^\dagger + u \chi^\dagger u \right) \bar{N} N \\ &= \frac{i c_1}{(2f)^4} \left\langle \left(\chi \frac{1}{4!} \pi^4 + \pi \chi \frac{1}{3!} \pi^3 + \frac{\pi^2}{2!} \chi \frac{\pi^2}{2!} + \frac{\pi^3}{3!} \chi \pi + \frac{\pi^4}{4!} \pi^4 \chi \right) \right. \\ & \quad \left. + \left(\chi^\dagger \frac{1}{4!} \pi^4 + \pi \chi^\dagger \frac{1}{3!} \pi^3 + \frac{\pi^2}{2!} \chi^\dagger \frac{\pi^2}{2!} + \frac{\pi^3}{3!} \chi^\dagger \pi + \frac{\pi^4}{4!} \pi^4 \chi^\dagger \right) \right\rangle \bar{N} N \\ &= \frac{i B_0 c_1}{6f^4} \langle \mathcal{M} \pi^4 \rangle = \frac{B_0 c_1}{6f^4} \left\langle \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \begin{pmatrix} \pi^0 + \eta/\sqrt{3} & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi^0 + \eta/\sqrt{3} \end{pmatrix}^4 \right\rangle \bar{N} N \\ &= \frac{i B_0 c_1}{6f^4} \left\langle \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \begin{pmatrix} (\pi^0 + \eta/\sqrt{3})^2 + 2\pi^+\pi^- & 2\sqrt{2/3}\pi^-\eta \\ 2\sqrt{2/3}\pi^+\eta & (-\pi^0 + \eta/\sqrt{3})^2 + 2\pi^+\pi^- \end{pmatrix}^2 \right\rangle \bar{N} N. \end{aligned} \quad (\text{A15})$$

Taking only the relevant terms, we obtain the $\eta\pi^0\pi^+\pi^-\bar{N}N$ vertex as

$$\begin{aligned} & \frac{i B_0 c_1}{6f^4} \left(m_u \times 8\eta\pi^0\pi^+\pi^-/\sqrt{3} - m_d \times 8\eta\pi^0\pi^+\pi^-/\sqrt{3} \right) \bar{N} N \\ &= \frac{i 4c_1 B_0}{3\sqrt{3}f^4} (m_u - m_d) \eta\pi^0\pi^+\pi^-\bar{N} N = -i \frac{4c_1 m_1^2}{3\sqrt{3}f^4} \eta\pi^0\pi^+\pi^-\bar{N} N. \end{aligned} \quad (\text{A16})$$

Thus, the $\eta\pi^0\pi^+\pi^-\bar{N}N$ vertex from the c_1 term is given as

$$g_{\eta\pi^0\pi^+\pi^-\bar{N}N}^{(c_1)} = -i \frac{4c_1 m_1^2}{3\sqrt{3}f^4}. \quad (\text{A17})$$

From Eq. (A15), the $\eta3\pi^0\bar{N}N$ vertex $g_{\eta3\pi^0\bar{N}N}^{(c_1)}$ is given as

$$g_{\eta3\pi^0\bar{N}N}^{(c_1)} = i \frac{B_0 c_1}{6f^4} \frac{4}{\sqrt{3}} (m_u - m_d) = -i \frac{2c_1 m_1^2}{3\sqrt{3}f^4}. \quad (\text{A18})$$

A.4.3. The vertex from the c_5 term

The $\eta\pi^0\pi^+\pi^-\bar{N}N$ vertex from the c_5 term is

$$\begin{aligned} & i c_5 \bar{N} \left(\frac{1}{(2f)^4} \left(\chi \frac{1}{4!} \pi^4 + \pi \chi \frac{1}{3!} \pi^3 + \frac{\pi^2}{2!} \chi \frac{\pi^2}{2!} + \frac{\pi^3}{3!} \chi \pi + \frac{\pi^4}{4!} \pi^4 \chi \right) \right. \\ & \quad \left. + \left(\chi^\dagger \frac{1}{4!} \pi^4 + \pi \chi^\dagger \frac{1}{3!} \pi^3 + \frac{\pi^2}{2!} \chi^\dagger \frac{\pi^2}{2!} + \frac{\pi^3}{3!} \chi^\dagger \pi + \frac{\pi^4}{4!} \pi^4 \chi^\dagger \right) \right) N - \frac{i c_5}{2} \left(-\frac{4m_1^2}{3\sqrt{3}f^4} \right) \bar{N} N \\ &= \frac{i 4B_0 c_5}{4! \cdot (2f)^4} \bar{N} \left(\mathcal{M} \pi^4 + 4\pi \mathcal{M} \pi^3 + 6\pi^2 \mathcal{M} \pi^2 + 4\pi^3 \mathcal{M} \pi + \pi^4 \mathcal{M} \right) N + \frac{i 2c_5 m_1^2}{3\sqrt{3}f^4} \bar{N} N. \end{aligned} \quad (\text{A19})$$

The terms contributing to the $\eta\pi^0\pi^+\pi^-\bar{N}N$ vertex are given as

$$\begin{aligned} & \frac{B_0c_5}{6f^4} \left(m_u(\bar{p}p) \left(\frac{8}{\sqrt{3}}\eta\pi^0\pi^+\pi^- \right) - m_d(\bar{n}n) \left(\frac{8}{\sqrt{3}}\eta\pi^0\pi^+\pi^- \right) \right) + \frac{i2c_5m_1^2}{3\sqrt{3}f^4}\bar{N}N \\ &= \frac{i4B_0c_5}{3\sqrt{3}f^4}(m_u\bar{p}p - m_d\bar{n}n)\eta\pi^0\pi^+\pi^- + \frac{i2c_5m_1^2}{3\sqrt{3}f^4}(\bar{p}p + \bar{n}n)\eta\pi^0\pi^+\pi^-. \end{aligned} \quad (\text{A20})$$

Thus, the $\eta\pi^0\pi^+\pi^-\bar{N}N$ vertex is obtained as

$$g_{\eta\pi^0\pi^+\pi^-\bar{N}N} = i \frac{4B_0c_5}{3\sqrt{3}f^4} \text{diag}(m_u, -m_d) + i \frac{2c_5m_1^2}{3\sqrt{3}f^4} \mathbf{1}. \quad (\text{A21})$$

From Eq. (A19), the $\eta 3\pi^0\bar{N}N$ vertex is given as

$$g_{\eta 3\pi^0\bar{N}N} = i \frac{2B_0c_5}{3\sqrt{3}f^4} \text{diag}(m_u, -m_d) + \frac{c_5m_1^2}{3\sqrt{3}f^4} \mathbf{1}. \quad (\text{A22})$$

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